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# NOTE ON THE SEMI-IMPLICIT INTEGRATION OF A FINE MESH LIMITED-AREA PREDICTION MODEL ON AN OFFSET GRID

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**ABSTRACT**

A primitive-equation fine mesh limited-area barotropic model has been integrated using time steps of 0.5 hr. The increased time step possible and the reduction in computation time are due to the implicit treatment of the terms governing gravity waves in the difference equations. Comparative integrations of the semi-implicit model and an explicit model show only minor differences in a 24-hr forecast, but the former achieves a time advantage of 3.5:1.

## 1. INTRODUCTION

Recently, Gerrity and McPherson (1969) published a paper concerning the integration of a fine mesh primitive-equation barotropic model over a limited portion of the Northern Hemisphere. This simple model was intended to serve as a steppingstone in the development of a high-resolution multilevel short-range prediction model, capable of being competitive in an operational environment. The experiments were successful in that 24-hr forecasts made from unbalanced initial data and with very simple boundary conditions exhibited some superiority over the quasi-hemispheric low-resolution barotropic and baroclinic models currently operational at the National Meteorological Center (NMC).

The numerical time integration technique used was an explicit one, the Euler-backward (Kurihara 1965). It has the desirable characteristic of damping high-frequency oscillations, but is costly to use. This disadvantage becomes much more serious as horizontal resolution is further increased, for the maximum time step allowable for computational stability is determined mainly by the velocity of gravity wave propagation and the distance between adjacent grid points. The present note describes a relaxation of this stringent stability criterion by use of a semi-implicit time integration scheme.

Implicit integration techniques have been examined by many investigators, including Kurihara (1965), but have not as yet been widely used in numerical weather prediction. In recent years, workers in the Soviet Union have experimented successfully with implicit techniques. Most prominent is Marchuk (1964, 1965) who, with his collaborators (Marchuk et al. 1967), has applied an implicit scheme to the integration of the primitive equations on a limited area.

Recently, Robert (1969) has applied an implicit method to the solution of a spectral model. More recently, Kwizak and Robert (1971) have developed a semi-implicit method for a finite difference primitive-equation barotropic model over a quasi-hemispheric domain. In their approach to the finite-difference model, gravitational oscillations are approximated implicitly, but rotational (meteorological) modes are treated explicitly. In this way, the maximum allowable time step is made to depend on the propagation speed of the meteorological modes rather than the gravity modes. The maximum time step is thus made comparable to that utilized in quasi-geostrophic models.

Section 2 describes a semi-implicit technique, very similar to Kwizak and Robert's, applied to a fine-mesh limited area primitive-equation barotropic model. A comparison of forecasts obtained by the implicit integration with those from an explicit integration is presented in section 4.

## 2. FORMULATION OF THE DIFFERENCE EQUATIONS

The primitive equations of motion and the equation of continuity for a homogeneous, incompressible fluid may be written in the "invariant" form (Shuman and Stackpole 1969) as

$$\frac{\partial u}{\partial t}+\frac{\partial \phi}{\partial x}-fv=\zeta v-\frac{\partial K}{\partial x}, \tag{1}$$

$$\frac{\partial v}{\partial t}+\frac{\partial \phi}{\partial y}+fu=-\zeta u-\frac{\partial K}{\partial y}, \tag{2}$$

and

$$\frac{\partial \phi}{\partial t}+gH_0m^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=-m^2\left[\frac{\partial(\phi u)}{\partial x}+\frac{\partial(\phi v)}{\partial y}\right]. \tag{3}$$

Here, $u$ and $v$ are scaled velocity components (velocities divided by the map factor $m$); the geopotential $\phi$ is a departure from a standard value ($gH_0$); and the relative vorticity $\zeta$ and the kinetic energy $K$ are defined by

$$\zeta=m^2\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right) \tag{4}$$

and

$$K=\frac{1}{2}\left[(mu)^2+(mv)^2\right]. \tag{5}$$

In transforming this system of differential equations to a system of difference equations, we will make use of the definitions

$$\overline{F}^{\tau}\equiv\frac{1}{2}\left[F\left(\tau+\frac{\Delta\tau}{2}\right)+F\left(\tau-\frac{\Delta\tau}{2}\right)\right]$$

and

$$F_\tau=\frac{1}{\Delta\tau}\left[F\left(\tau+\frac{\Delta\tau}{2}\right)-F\left(\tau-\frac{\Delta\tau}{2}\right)\right]. \tag{6}$$

We will formulate the difference equations on an offset grid lattice; that is, the geopotential departures $\phi$ will be defined at the intersections of a regular lattice (grid points), and the velocity components will be defined at the centers of squares, the corners of which are grid points. The network of grid points is defined by

$$x=j\Delta x,\ j=0,1,2,\ .\ .\ .\ ,M;\ y=k\Delta y,\ k=0,1,2,\ .\ .\ .\ ,N.$$

The centers of the boxes are defined by

$$x=(j+\tfrac{1}{2})\Delta x,\ j=0,1,2,\ .\ .\ .\ ,M-1;\quad y=(k+\tfrac{1}{2})\Delta y,\ k=0,1,2,\ .\ .\ .\ ,N-1.$$

The offset grid arrangement offers several advantages. It avoids the possibility of spatial computational modes arising from the use of customary centered-difference approximations, as discussed by Platzman (1958) and Matsuno (1966). Truncation error is also reduced in the linear terms, relative to that associated with standard centered-differences, since here differences are taken over only one grid increment. Finally, the offset grid allows a convenient form of the finite-difference Laplacian operator to be used in the implicit scheme, as will be shown presently.

An explicit difference analog of the system (1–3) can be written as

$$\overline{u}_t^{t}+\overline{\phi}_x^{y}-f_0v=\overline{\overline{v}^{xy}(\zeta+f')-\overline{K}_x^{y}}^{xy}, \tag{7}$$

$$\overline{v}_t^{t}+\overline{\phi}_y^{x}+f_0u=-\overline{\overline{u}^{xy}(\zeta+f')-\overline{K}_y^{x}}^{xy}, \tag{8}$$

and

$$\overline{\phi}_t^{t}+gH_0m^2(\overline{u}_x^{y}+\overline{v}_y^{x})=-m^2[(\overline{\phi}^{xyy}\overline{u}^{y})_x+(\overline{\phi}^{xxy}\overline{v}^{x})_y], \tag{9}$$

where

$$\zeta=\overline{m^2}^{xy}(\overline{v}_x^{y}-\overline{u}_y^{x}) \tag{10}$$

and $f=f_0+f'$, $f_0$ being the value of the Coriolis parameter at 45°N. For introducing the semi-implicit character to the difference equations, we average in time those terms that principally govern the gravitational oscillations: the pressure gradient terms in the momentum equations and the linear divergence term in the continuity equation. Additionally, we average in time the "linear" part of the Coriolis terms ($f_0v$ and $f_0u$), so that the truncation error in the Coriolis and pressure gradient terms will be similar. This time-average operator is defined simply as

$$\overline{F}^{2t}\equiv\frac{1}{2}\,[F(t+\Delta t)+F(t-\Delta t)].$$

Equations (7–9) become

$$\overline{u}_t^{t}+\overline{(\overline{\phi}_x^{y})}^{2t}-f_0\overline{v}^{2t}=\overline{\overline{v}^{xy}(\zeta+f')-\overline{K}_x^{y}}^{xy}, \tag{11}$$

$$\overline{v}_t^{t}+\overline{(\overline{\phi}_y^{x})}^{2t}+f_0\overline{u}^{2t}=-\overline{\overline{u}^{xy}(\zeta+f')-\overline{K}_y^{x}}^{xy}, \tag{12}$$

and

$$\overline{\phi}_t^{t}+gH_0m^2\overline{(\overline{u}_x^{y}+\overline{v}_y^{x})}^{2t}=-m^2[(\overline{\phi}^{xyy}\overline{u}^{y})_x+(\overline{\phi}^{xxy}\overline{v}^{x})_y]. \tag{13}$$

Equations (11–13) can be rewritten, expressing values at future times in terms of values at the present and past times:

$$u^{n+1}+\Delta t(\overline{\phi}_x^{y})^{n+1}-f_0\Delta tv^{n+1}=u^{n-1}-\Delta t(\overline{\phi}_x^{y})^{n-1}+f_0\Delta tv^{n-1}+2\Delta t[\overline{\overline{v}^{xy}(\zeta+f')-\overline{K}_x^{y}}^{xy}]^n \tag{14}$$

and

$$v^{n+1}+\Delta t(\overline{\phi}_y^{x})^{n+1}+f_0\Delta tu^{n+1}=v^{n-1}-\Delta t(\overline{\phi}_y^{x})^{n-1}-f_0\Delta tu^{n-1}-2\Delta t[\overline{\overline{u}^{xy}(\zeta+f')+\overline{K}_y^{x}}^{xy}]^n, \tag{15}$$

valid in grid "boxes" $[j\pm(1/2),\ k\pm(1/2)]$ and at grid points $(j,k)$,

$$\phi^{n+1}+gH_0m^2\Delta t(\overline{u}_x^{y}+\overline{v}_y^{x})^{n+1}=\phi^{n-1}-gH_0m^2\Delta t(\overline{u}_x^{y}+\overline{v}_y^{x})^{n-1}-2\Delta tm^2[(\overline{\phi}^{xyy}\overline{u}^{y})_x+(\overline{\phi}^{xxy}\overline{v}^{x})_y]^n. \tag{16}$$

Henceforth, the right sides of eq (14–16) will be denoted by $A$, $B$, and $C$, respectively, If one eliminates $v^{n+1}$ from eq (14) and $u^{n+1}$ from eq (15), the resulting equations are

$$u^{n+1}=\hat{A}-\frac{\Delta t}{1+(f_0\Delta t)^2}\,[(\overline{\phi}_x^{y})^{n+1}+f_0\Delta t(\overline{\phi}_y^{x})^{n+1}] \tag{17}$$

and

$$v^{n+1}=\hat{B}-\frac{\Delta t}{1+(f_0\Delta t)^2}\,[(\overline{\phi}_y^{x})^{n+1}-f_0\Delta t(\overline{\phi}_x^{y})^{n+1}], \tag{18}$$

where

$$\hat{A}=A+f_0\Delta tB \text{ and } \hat{B}=B-f_0\Delta tA.$$

Forming the divergence from eq (17) and (18) and substituting the result into the left side of (16) yields a Helmholtz-type equation in $\phi^{n+1}$:

$$\phi^{n+1}-\gamma\nabla^2\phi^{n+1}=C-gH_0m^2\Delta t(\overline{\hat{A}}_x^{y}+\overline{\hat{B}}_y^{x}), \tag{19}$$

where

$$\gamma=2\left(\frac{\Delta t}{2\Delta x}\right)^2\Big/[1+(f_0\Delta t)]^2,$$

and the symbol $\nabla^2$ denotes a finite-difference Laplacian operator defined by

$$\nabla^2\phi_{j,k}\equiv\phi_{j+1,k+1}+\phi_{j+1,k-1}+\phi_{j-1,k-1}+\phi_{j-1,k+1}-4\phi_{j,k}. \tag{20}$$

Equations (17–19) constitute the semi-implicit formulation of the model. Equation (19) can be solved for $\phi_{j,k}^{n+1}$, given appropriate boundary values, by any one of a variety of numerical techniques. Once $\phi_{j,k}^{n+1}$ has been obtained for all grid points, the velocity components in the "boxes" can be obtained from eq (17) and (18).

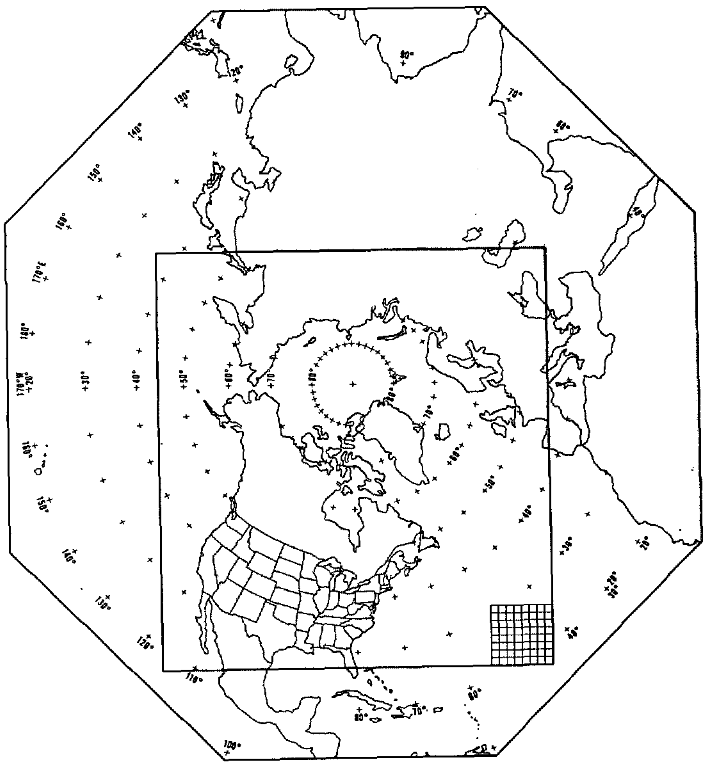

FIGURE 1.—Polar stereographic map of the Northern Hemisphere showing the octagon within which data were available. Also shown is the rectangular boundary of the limited-area fine mesh within which the model equations were solved. In the lower right-hand part of the rectangle, a portion of the fine-mesh grid is illustrated.

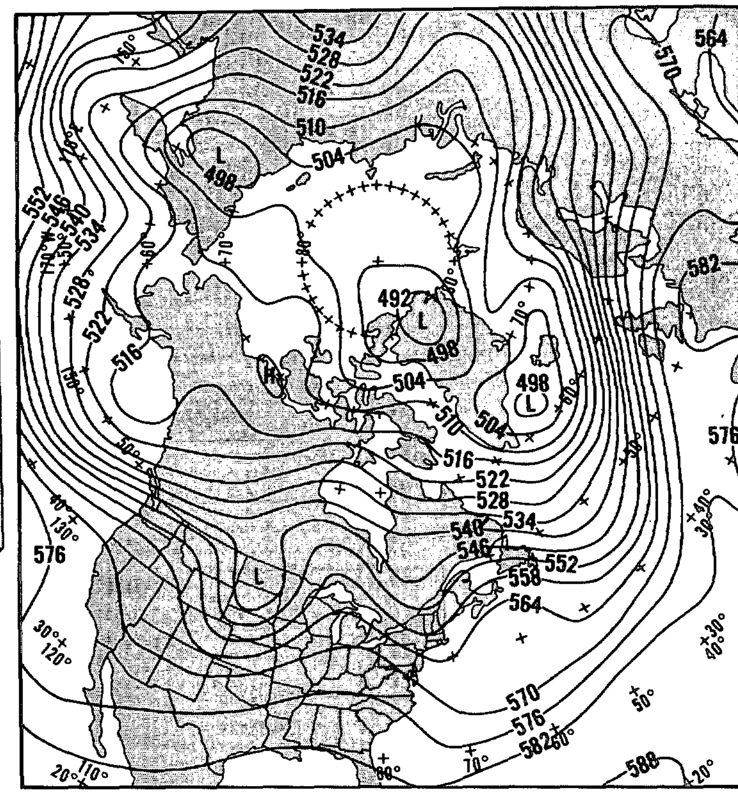


FIGURE 2.—Initial configuration of the 500-mb geopotential surface after interpolation to the fine mesh. The contours are labeled in dekameters. This analysis is for 1200 GMT on Mar. 27, 1968.

## 3. BOUNDARY CONDITIONS AND INITIAL CONDITIONS

In the previously mentioned experiments (Gerrity and McPherson 1969) with an explicit primitive-equation fine mesh limited-area model, temporally constant lateral boundaries were employed. Such a specification has also been used by the Russian workers in limited-area modeling (Marchuk et al. 1967) and in modified form by the British (Bushby and Timpson 1967, Bushby 1969). The behavior of gravitational oscillations on a temporally constant boundary has been the subject of a recent analysis by Gerrity and McPherson (1970). Accordingly, the lateral boundaries in the present experiments were also held constant with time. Specifically, geopotentials at the outer row/column of the grid lattice, and the velocity components in the outermost row/column of boxes are invariant with time. It was observed in early experiments that, where the boundaries are parallel and close to a strong jet, annoying distortions of the fields eventually developed. This was rectified by applying what is effectively the first step of the Lax-Wendroff method (Richtmyer 1962) *only* at the first interior grid points and boxes. That is, the leading terms on the right sides of eq (14–16) were replaced by their averages over the neighboring four points. This constitutes a rather strong diffusion in the zone immediately adjacent to the boundaries.

Initial data for this experiment were NMC operational analyses of 500-mb geopotential slightly modified to satisfy the ellipticity criterion; nondivergent winds were obtained by solving the nonlinear balance equation (Shuman 1957) on a quasi-hemispheric polar stereographic grid of standard mesh size, 381 km at 60°N. These coarse-mesh geopotentials and winds were then interpolated biquadratically to the fine-mesh grid points in the limited area shown in figure 1. The balance achieved on the coarse-mesh grid is not seriously disturbed by the interpolation to the fine-mesh grid.

## 4. RESULTS

The semi-implicit model was integrated over the limited area shown in figure 1, using the same test case, 1200 GMT on Mar. 27, 1968, as in the previous paper (Gerrity and McPherson 1969). The initial data were modified as indicated in the previous paragraph; the initial 500-mb height field is shown in figure 2. Time steps of 0.5 hr easily sufficed to satisfy the linear stability criteria associated with the meteorological modes; the horizontal mesh length (distance between adjacent values of the same parameter) is 190.5 km at 60°N.

For comparison, the explicit model reported in the previous paper was reintegrated, using the modified initial data. A 5-min time step satisfied the linear stability criteria.

Figures 3 through 6 display the resulting 12- and 24-hr forecasts from the semi-implicit and explicit integrations.

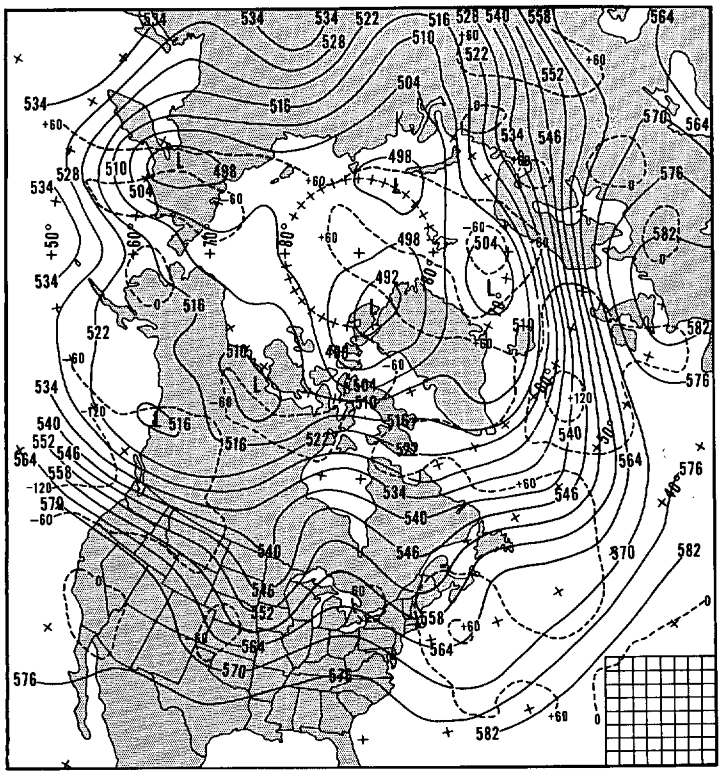

FIGURE 3.—The 500-mb geopotential height field as predicted to verify 12 hr after the initial time by the semi-implicit model with a 0.5-hr time step. The dashed lines represent errors (forecast minus observed) in meters.

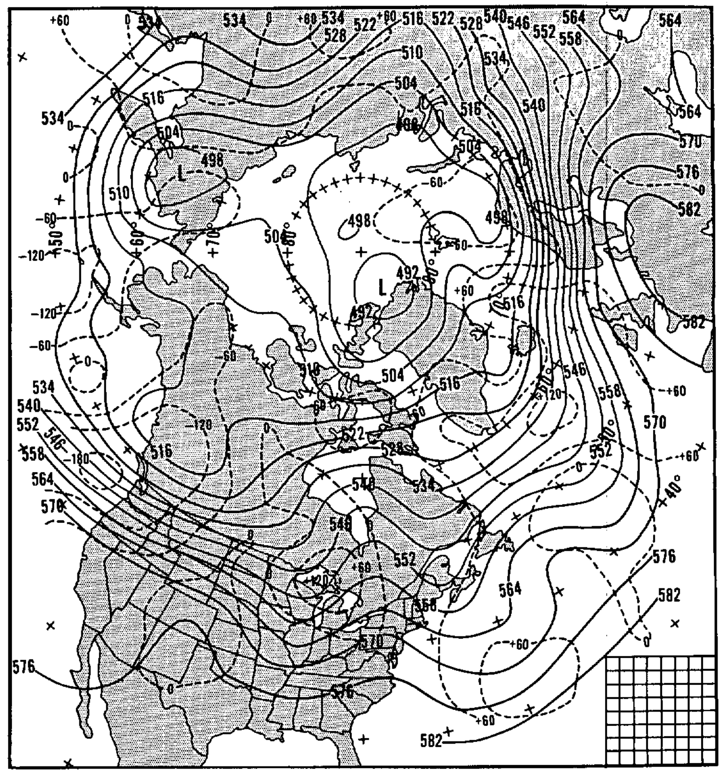

FIGURE 5.—The 500-mb geopotential height field as predicted to verify 24 hr after the initial time by the semi-implicit model with a 0.5-hr time step.

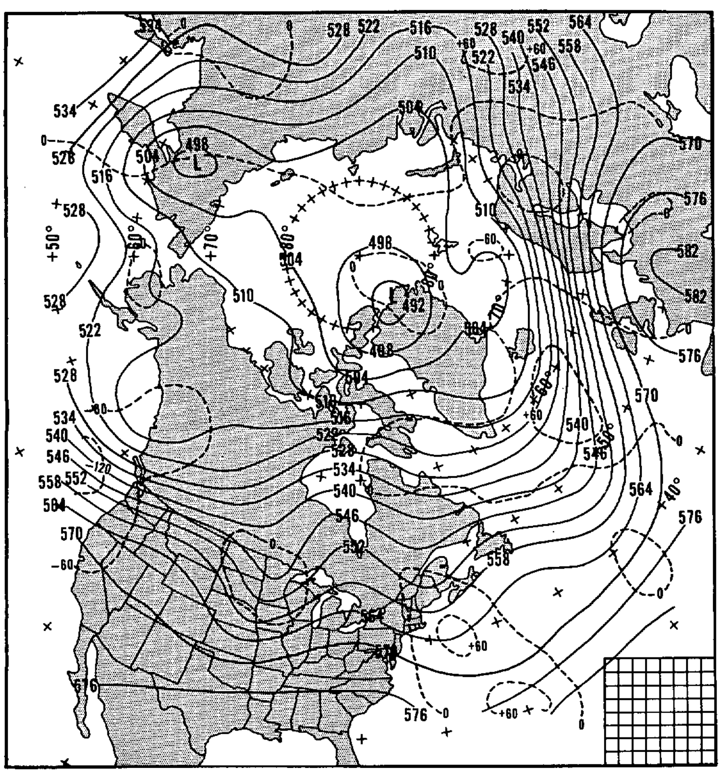

FIGURE 4.—The 500-mb geopotential height field as predicted to verify 12 hr after the initial time by the explicit model with a 5-min time step.

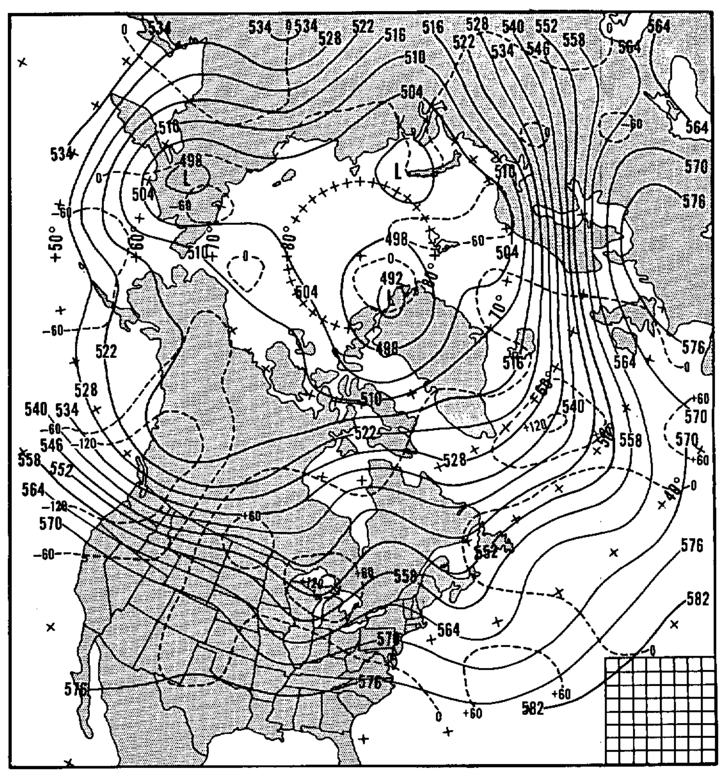

FIGURE 6.—The 500-mb geopotential height field as predicted to verify 24 hr after the initial time by the explicit model with a 5-min time step.

In each figure, the error (forecast minus observed) is indicated by dashed lines. Comparison of figures 3 and 4, the 12-hr semi-implicit and explicit forecasts, respectively, indicates that the forecasts are very similar. The major features are treated in approximately the same manner. The most noticeable differences can be seen in the error fields, especially over the northwestern part of the United States and southwestern Canada and over the north Atlantic. The 24-hr forecasts shown in figures 5 and 6 also display relatively minor differences.

It should be noted that the comparison effected in figures 3–6 is not simply one of explicit versus implicit integration schemes. There are a number of other differences in the formulation of the two models; for example, the explicit model is formulated on a lattice of regular points rather than an offset lattice, and consequently dissimilar difference approximations are made in the nonlinear terms. It is, therefore, not surprising that there are differences in the forecasts.[1]

Perhaps the most important difference, however, is in the amount of computation time required. The explicit 24-hr forecast, using a 5-min time step, required 455 s on a CDC 6600 computer; the semi-implicit forecast, using a 0.5-hr time step, required 130 s. The time advantage achieved by use of the semi-implicit method is approximately a factor of 3½; it should be pointed out that the Helmholtz-type eq (19) was solved using standard relaxation techniques without much effort applied to obtaining an efficient solution. Kwizak and Robert (1971), with more care, have improved this to a factor of 4.

## 5. SUMMARY

A semi-implicit integration technique has been developed for a primitive-equation barotropic model, applied to a limited portion of the atmosphere. The resulting forecasts are of approximately the same quality as those obtained by explicit methods. The principal benefit, an advantage in computing time of 3.5:1, is of major significance.

It seems clear that the development of an operationally feasible short-range weather prediction model must utilize this advantage, since such a model will of necessity incorporate relatively great resolution both horizontally and vertically. Accordingly, an effort is underway to apply a semi-implicit technique to a multilevel model.

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The work reported here is one segment of a joint effort on the part of the author and Dr. J. P. Gerrity, Jr., of NMC to develop a high-resolution short-range operational prediction model. In large measure, this note is also a contribution of Dr. Gerrity.

[1] In a separate experiment, not included here, the semi-implicit model was integrated over a quasi-hemispheric rectangle circumscribing the octagon shown in figure 1, with a mesh length of 381 km at 60°N and a time step of 1 hr. Equations (7–9), a direct explicit analog, were also integrated over the same domain using a 10-min time step. The resulting comparison of explicit versus semi-implicit integration indicated no height difference greater than 10 m after 24 hr. A similar experiment, but over the limited area, revealed somewhat larger differences as much as 25 m at a few points after 24 hr. This is probably related to the lateral boundaries being located in synoptically active regions in the latter experiment.

## REFERENCES

Bushby, F. H., "Further Developments of a Model for Forecasting Rain and Weather," *Proceedings of the WMO/IUGG Symposium on Numerical Weather Prediction, Tokyo, Japan, November 26–December 4, 1968*, Japan Meteorological Agency, Tokyo, Mar. **1969**, pp. II–75—II–84.

Bushby, F. H., and Timpson, Margaret S., "A 10-Level Atmospheric Model and Frontal Rain," *Quarterly Journal of the Royal Meteorological Society*, Vol. 93, No. 395, London, England, Jan. **1967**, pp. 1–17.

Gerrity, Joseph P., Jr., and McPherson, Ronald D., "Development of Limited Area Fine-Mesh Prediction Model," *Monthly Weather Review*, Vol. 97, No. 9, Sept. **1969**, pp. 665–669.

Gerrity, Joseph P., Jr., and McPherson, Ronald D., "Noise Analysis of a Limited-Area Fine-Mesh Prediction Model," *ESSA Technical Memorandum* WBTM NMC 46, National Meteorological Center, Suitland, Md., Feb. **1970**, 81 pp.

Kurihara, Yoshio, "On the Use of Implicit and Iterative Methods for the Time Integration of the Wave Equation," *Monthly Weather Review*, Vol. 93, No. 1, Jan. **1965**, pp. 33–46.

Kwizak, Michael, and Robert, André J., "A Semi-Implicit Scheme for Grid Point Atmospheric Models of the Primitive Equations," *Monthly Weather Review*, Vol. 99, No. 1, Jan. **1971**, pp. 32–36.

Marchuk, Gurii I., "Teoreticheskaia Model' Prognoza Pogody" (Theoretical Model of a Weather Forecast), *Doklady*, Vol. 155, No. 5, Akademiia Nauk SSSR, Moscow, Apr. 11, **1964**, pp. 1062–1065.

Marchuk, Gurii I., *Numerical Methods for Solving Weather-Forecasting and Climate-Theory Problems*, Pt. I, Research Translation T–R–520, U.S. Air Force Cambridge Research Laboratories, Bedford, Mass., Nov. **1965**, 101 pp.

Marchuk, Gurii I., Kontarev, G. R., and Rivin, G. S., "Kratkosrochnyĭ Prognoz Pogody Po Polnym Uravneniiam na Ogranichennoĭ Territorii" (Short Range Weather Forecasting Using Primitive Equations for a Restricted Region), *Izvestiia, Atmosfery i Okeana Fizika*, Vol. 3, No. 11, Akademiia Nauk SSSR, Moscow, Nov. **1967**, pp. 1166–1178.

Matsuno, Taroh, "False Reflection of Waves at the Boundary Due to the Use of Finite Differences," *Journal of the Meteorological Society of Japan*, Ser. 2, Vol. 44, No. 2, Tokyo, Apr. **1966**, pp. 145–157.

Platzman, George W., "The Lattice Structure of the Finite-Difference Primitive and Vorticity Equations," *Monthly Weather Review*, Vol. 86, No. 8 Aug. **1958**, pp. 285–292.

Richtmyer, Robert D., "A Survey of Difference Methods for Non-Steady Fluid Dynamics," *NCAR Technical Notes* 63–2, National Center for Atmospheric Research, Boulder, Colo., **1962**, 25 pp.

Robert, André J., "The Integration of a Spectral Model of the Atmosphere by the Implicit Method," *Proceedings of the WMO/IUGG Symposium on Numerical Prediction, Tokyo, Japan, November 26–December 4, 1938*, Japan Meteorological Agency, Tokyo, Mar. **1969**, pp. VII–19–VII–24.

Shuman, Frederick G., "Numerical Methods in Weather Prediction: I. The Balance Equation," *Monthly Weather Review*, Vol. 85, No. 10, Oct. **1957**, pp. 329–332.

Shuman, Frederick G., and Stackpole, John D., "The Currently Operational NMC Model, and Results of a Recent Simple Numerical Experiment," *Proceedings of the WMO/IUGG Symposium on Numerical Prediction, Tokyo, Japan, November 26–December 4*, 1968, Japan Meteorological Agency, Tokyo, Mar. **1969**, pp. II–85—II–98.

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